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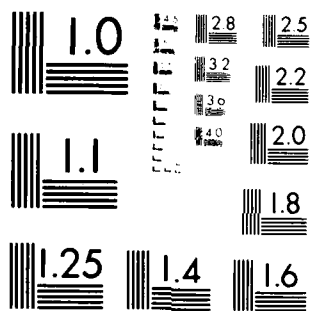
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20. Abstract (continued)

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## Some Remarks on Rate-Dependent Plasticity

by

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Abstract. First, some motivation is provided for the development of a relatively simple rate-dependent theory of plasticity based on an elastic-viscoplastic constitutive model. This idealization includes unloading and allows for suitable definition of plastic strain. Next, within the scope of a purely mechanical theory, a special constitutive response is used to discuss the nature of constitutive restrictions for a finitely deforming rate-dependent material, which are obtained from an appropriate work inequality. With the help of these restrictions, certain features of the theory are elaborated upon with particular reference to normality and convexity.



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## 1. Introduction

This paper elaborates on certain aspects of rate-dependent plasticity in the presence of finite deformation, which include as a special case the corresponding results for a rate-independent theory. The discussion is carried out on this occasion entirely within the scope of the purely mechanical theory, leaving aside relevant thermodynamical aspects of the subject.

By way of background, it should be recalled here that in the usual theory of rate-independent plasticity, the rate of plastic strain and the rate of work-hardening are expressed as linear functions of the rate of strain or the rate of stress. The coefficients of response functions in these expressions, as well as the loading functions and other constitutive response functions, are all independent of the rate of strain, the rate of stress and time derivatives of other kinematical ingredients. Indeed, by the very nature of the idealization of rate-independent theory, the time rate of quantities are independent of the time scale used to compute the rate of change so that, for example, the plastic strain rate is homogeneous of degree one in the rate of strain (or the rate of stress). In fact, in the rate-independent theory, the time variable may be interpreted as any parameter which is monotonically increasing with time during deformation. By contrast, a rate-dependent theory of plasticity is intended to characterize "rate-dependent" behavior by including rate quantities in the constitutive response functions and possibly also in the yield or loading functions.

For definiteness, consider the response of a rate-dependent material (say that of a typical ductile metal) in a one-dimensional test -- either in tension or compression -- in which the strain may be moderately large or even large. Let  $e$  and  $s$  stand, respectively, for the component  $e_{11}$  of the Lagrangian strain tensor and the component  $s_{11}$  of the symmetric Piola-Kirchhoff stress tensor. Figure 1 shows the familiar plot of stress versus

strain of an elastic-viscoplastic material\* at various strain rates  $\dot{\epsilon}_1, \dot{\epsilon}_2, \dots$ , where a superposed dot designates differentiation with respect to time  $t$ . Also shown in Fig. 1 is the corresponding rate-independent idealization represented by the curve OAB, where in the linear range OA the material is elastic. The point A representing the elastic limit is assumed to be coincident with the initial yield.

Rate-dependent behavior of materials encompasses an important and difficult chapter in the theory of inelastic behavior of materials. Consider the enormous difficulties one encounters in attempting to include all relevant features in the development of rate-independent theory of plasticity, where all time effects are ignored. Given this, the further difficulties of rate-dependent theory is evident. The terms "rate-dependent" or "rate-sensitive" behavior of materials, frequently used in the literature, refer loosely to various developments intended to reflect a variety of material behavior. For example, elastic-viscoplastic or viscoelastic-plastic theories, or still other rate-dependent theories of material behavior, although difficult to classify, include some dependence on the rate of strain. One particular category which has attracted considerable attention from time to time is one in which the theory attempts to describe material behavior during loading with the initial response prior to yield regarded as elastic, but usually such theories do not deal with the problem of unloading. In such rate-dependent theories an important feature which seems to have been dealt with from various points of view is the prediction of an increase in the linear range of the stress-strain response with increasing strain rate which, in turn, leads to the prediction of higher initial yield stress at higher strain rates. The interpretations of experimental results in this area seem inconclusive and appear to lack any discussion about the nature of unloading and a suitable

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\* Although the stress-strain responses shown in Fig. 1 are plotted in the  $s$ - $e$  plane, the corresponding plots of engineering stress versus engineering strain or true stress versus true strain exhibit similar features.

definition for plastic strain in rate-dependent materials. For general background information on the subject reference may be made to an article by Perzyna [1] and to a recent experimental paper by Klepaczko and Duffy [2], where a large number of additional references are cited.

An important aspect of the development of theories of the type under discussion involves obtaining realistic material response -- in accordance with experimental observations -- by imposing physically plausible restrictions on the constitutive equations. For example, such restrictions may be effected by some appropriate statement (or statements) of the Second Law of Thermodynamics, or by an appeal to certain stability criteria such as Hadamard's stability condition. In this connection, a fruitful idea involving the notion of nonnegative work in a closed stress cycle was advanced by Drucker [3] in 1951; and, with the limitation to small deformation, it was expressed in the context of the purely mechanical, rate-independent theory of elastic-plastic materials. This idea, again in the presence of small deformation, was subsequently extended by Drucker [4] to rate-dependent theory and was referred to by him as "stability postulate." A related postulate was introduced in 1961 by Il'iushin [5] in the context of the linearized theory of plasticity with small strain. The latter again involves a work inequality, which is defined over a closed strain cycle; and, as noted in [5], is less restrictive than Drucker's postulate [3]. Mention should also be made of related papers by Drucker [6,7] and by Palmer et al. [8], which contain some discussion pertaining to the normality of plastic strain rate and convexity of the loading surfaces in the presence of small deformation.

More recently, in the context of finite deformation and in a strain space setting, Naghdi and Trapp [9] introduced a physically plausible work assumption and used this primitive assumption to derive a work inequality.



The work inequality was then used to obtain constitutive restrictions in the rate-independent theory of elastic-plastic materials [9,10]. The work inequality derived in [9], upon specialization to small deformation, may be compared with the postulates of Il'iushin and Drucker (see [9, section 3]). Moreover, it is clear from the derivation of the work inequality in [9] that its validity is not limited to rate-independent plasticity but, in fact, is valid for a fairly large class of materials including those which may be rate-dependent.

In the present paper, after some preliminary background information pertaining to both "rate-independent" and "rate-dependent" inelastic behavior of materials, the main features of a model for a class of "rate-dependent" materials introduced previously in [11] are recalled in section 2. The constitutive response of this relatively simple model, which characterizes an elastic-viscoplastic behavior, is capable of describing material response during both loading and unloading and accommodates a suitable definition for plastic strain. Next, using a special constitutive equation for the stress response (see Eqs. (2.17a,b)) and the work inequality derived in [9], constitutive restrictions are derived in section 3. A part of these results may be regarded as a special case of those obtained in [11] with the use of fairly general constitutive equations. Most of the mathematical details of the developments in section 3 leading to the inequality (3.6) are placed in Appendix A at the end of the paper. Further, as discussed in section 4, the restrictions derived in section 3 bear on normality of certain expressions involving plastic strain rate and on convexity of loading surfaces in strain space. Also included in section 4 is a geometrical interpretation of the one-dimensional version of the main work inequality over a strain cycle utilized in section 3.

## 2. Preliminaries. General background.

Throughout the paper, we use both the standard direct and the Cartesian tensor notations. Often, however, we write the various expressions in component forms and display vector and tensor fields in terms of their rectangular Cartesian components. All subscripts take the values 1,2,3, and the usual summation convention is employed over repeated subscripts.

Let the motion of a body be referred to a fixed system of rectangular Cartesian axes and let the position of a typical particle in the present configuration at time  $t$  be designated by  $\underline{x}$  with rectangular Cartesian components  $x_i$ , where  $x_i = x_i(X_A, t)$  and  $X_A$  is a reference position of the particle. Further, let  $\underline{E}$  with rectangular Cartesian components  $e_{KL}$  be the symmetric Lagrangian strain defined by

$$e_{KL} = \frac{1}{2}(F_{iK}F_{iL} - \delta_{KL}) \quad , \quad F_{iK} = \frac{\partial x_i}{\partial X_K} \quad , \quad (2.1)$$

where  $F_{iK}$  are the components of the deformation gradient relative to the reference position and  $\delta_{KL}$  is the Kronecker symbol. We also recall the relationship between the nonsymmetric Piola-Kirchhoff stress tensor  $\underline{P}$  and the symmetric Piola-Kirchhoff stress tensor  $\underline{S}$ , namely

$$P_{iK} = F_{iL}S_{LK} \quad , \quad S_{LK} = S_{KL} \quad , \quad (2.2)$$

where  $p_{iK}$  and  $s_{KL}$  are, respectively, the rectangular Cartesian components of  $\underline{P}$  and  $\underline{S}$ .

For purposes of comparisons with later developments, we now summarize the main ingredients of a rate-independent theory of a finitely deforming elastic-plastic solid and base our results on the purely mechanical aspects of the subject contained in the papers of Green and Naghdi [12,13] and Naghdi and Trapp [14]. Thus, in addition to the strain tensor  $e_{KL}$ , we

assume the existence of a symmetric second order tensor-valued function  $\underline{E}^P$ , with rectangular Cartesian components  $e_{KL}^P(X_A, t)$ , called the plastic strain at  $X_A$  and  $t$ , and a scalar-valued function  $\kappa = \kappa(X_A, t)$  called a measure of work-hardening. It is assumed that the stress  $\underline{S}$  is given by the constitutive equation

$$s_{KL} = \hat{s}_{KL}(u) \quad , \quad u = (e_{MN}, e_{MN}^P, \kappa) \quad (2.3)$$

and that for fixed values of  $\underline{E}^P$  and  $\kappa$ , (2.3)<sub>1</sub> possesses an inverse of the form

$$e_{KL} = \hat{e}_{KL}(v) \quad , \quad v = (s_{MN}, e_{MN}^P, \kappa) \quad (2.4)$$

We use a strain space formulation of plasticity\* and, as in the paper of Casey and Naghdi [16], also regard the loading criteria of the strain space formulation as primary. The associated loading conditions in stress space can then be derived with the use of constitutive equations for stress. Thus, we admit the existence of a continuously differentiable scalar-valued yield (or loading) function  $g(u)$  such that, for fixed values of  $\underline{E}^P$  and  $\kappa$ , the equation

$$g(u) = 0 \quad (2.5)$$

represents a closed orientable hypersurface  $\partial E$  of dimension five enclosing an open region  $E$  of strain space. The function  $g$  is chosen so that  $g(u) < 0$  for all points in the region  $E$ . The hypersurface  $\partial E$  is called the yield (or loading) surface in strain space. The constitutive equations for  $\dot{\underline{E}}^P$  and  $\dot{\kappa}$  are [14]:

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\*Advantages of a strain space formulation of plasticity have been emphasized in [14, 15].

$$\dot{e}_{KL}^p = \begin{cases} 0 & \text{if } g < 0, & (a) \\ 0 & \text{if } g = 0 \text{ and } \hat{g} < 0, & (b) \\ 0 & \text{if } g = 0 \text{ and } \hat{g} = 0, & (c) \\ \lambda \rho_{KL} \hat{g} & \text{if } g = 0 \text{ and } \hat{g} > 0, & (d) \end{cases} \quad (2.6)$$

and

$$\dot{\kappa} = C_{KL} \dot{e}_{KL}^p, \quad (2.7)$$

where  $C_{KL}$  is a symmetric tensor-valued function of the variables  $U$ , a superposed dot denotes material time differentiation,

$$\hat{g} = \frac{\partial g}{\partial e_{MN}} \dot{e}_{MN} \quad (2.8)$$

and where  $\lambda$  and  $\rho_{KL}$  are, respectively, a scalar-valued and a symmetric tensor-valued function of  $U$ . The conditions involving  $g$  and  $\hat{g}$  in (2.6) are the loading criteria of the strain space formulation. Using conventional terminology, these four conditions in the order listed correspond to (a) an elastic state (or point in strain space); (b) unloading from an elastic-plastic state, i.e., a point in strain space for which  $g=0$ ; (c) neutral loading from an elastic-plastic state; and (d) loading from an elastic-plastic state. We assume that the coefficient of  $\hat{g}$  in (2.6d) is nonzero on the yield surface and, without loss in generality, we then set  $\rho_{KL} \neq 0$ ,  $\lambda > 0$ . The so-called "consistency" condition, namely  $\dot{g}=0$ , yields the relationship

$$1 + \lambda \rho_{KL} \left( \frac{\partial g}{\partial e_{KL}^p} + \frac{\partial g}{\partial \kappa} C_{KL} \right) = 0 \quad (2.9)$$

at all points on the yield surface through which loading can occur.

For a given loading function  $g(U)$ , with the aid of  $(2.4)_1$ , we can obtain a corresponding function  $f(V)$  by means of the formula

$$g(u) = g(e_{KL}(v), e_{KL}^p, \kappa) = f(v) \quad (2.10)$$

Because of the assumed smoothness of (2.4)<sub>1</sub>, for fixed values of  $\underline{E}^p$  and  $\kappa$ , the equation

$$f(v) = 0 \quad (2.11)$$

represents a hypersurface  $\partial S$  in stress space having the same geometrical properties as the hypersurface  $\partial E$  in strain space. The region enclosed by  $\partial S$  is denoted by  $S$ . It is clear from (2.10) that a point in strain space belongs to the region  $E$  (i.e.,  $g(u) < 0$ ) if and only if the corresponding point in stress space satisfies  $f(v) < 0$  and hence belongs to  $S$ . With the aid of the constitutive equation (2.4)<sub>1</sub>, loading conditions for stress space can now be derived from the loading criteria of strain space in (2.6) as discussed by Casey and Naghdi [16]. We do not record these conditions as they are not needed for the particular development of the present paper.

We are concerned in the present paper with a discussion of certain features of a relatively simple rate-dependent theory of plasticity. In order to motivate the subsequent developments, we begin by examining briefly the familiar idealized one-dimensional response of test results, especially since many concepts concerning mechanical behavior of materials are extensions of observations made in simple tension or simple compression or simple shear. Both Figs. 1 and 2 depict certain idealized response of materials. In both figures the curve OAB represents the so-called "static" response corresponding to an idealized rate-independent model. With reference to the curve OAB, we note that from the point O to the proportional limit A the material is linearly elastic and, since the deformation in this range is reversible, unloading takes place along AO. For loading above A (regarded to be coincident with the yield point), the deformation is irreversible and the

rate-independent material strain hardens along AQB. Unloading from a point such as Q is assumed to take place elastically along QR, which is generally taken to be parallel to OA. There are other features associated with the rate-independent response such as the lower compressive yield limit, but these need not be discussed here.

It is reasonable to assume that for sufficiently low strain rate, i.e., as the strain rate  $\dot{\epsilon}$  tends to zero (using the notation of section 1), the rate-dependent response of the material approaches that of the rate-independent model represented by OAB in Fig. 1. At higher rates of strain, the rate-dependent response of the material would be represented by  $OAB_1$  when the strain rate is  $\dot{\epsilon}_1$  and by  $OAB_2$  when the strain rate is  $\dot{\epsilon}_2 (> \dot{\epsilon}_1)$  and so on. Again, with reference to Fig. 1, consider a point P on the response curve  $OAB_2$  and through P draw a dashed line parallel to the s-axis intersecting the curve OAB at Q. Let  $e$  and  $e_p$  denote, respectively, the total strain and the plastic strain at Q on the rate-independent response curve OAB. Then, as indicated in Fig. 1, the segment OR represents the plastic strain at Q. In general, deceleration from a point P on the curve  $OAB_2$  would proceed along the dashed curves such as  $PQ_1, PQ_2$  to the right of PQ and such processes would be represented by dashed curves which end on the curve OAB representing the rate-independent response. In fact, as the process of deceleration takes place at a faster rate, in the limit one would approach the dashed line PQ. We may refer to PQ as a rapid path using an earlier terminology introduced in [17]. Clearly, every point along the line PQ experiences the same total strain  $e$  and the same amount of plastic strain  $e_p$  and this notion of rapid path enables one to define plastic strain for rate-dependent materials. Again, with reference to Fig. 1, reloading from a point along RQ at a given rate of strain (say for example at the rate  $\dot{\epsilon}_2$ ) would proceed along RQ until one reaches Q and then, consistent with the

idealization indicated, there has to be an allowance for jump in the value of the stress in order to reach the point P. Thereafter, as indicated in Fig. 1, further plastic deformation at the rate  $\dot{\epsilon}_2$  will continue along  $PB_2$ .

Experimental results of a fairly large class of rate-dependent materials often exhibit the phenomenon that at higher rates of strain the value of yield stress, sometimes referred to as "dynamic yield" stress, occurs at a value that is higher than the corresponding "static yield" of the rate-independent theory. Such observations may be accommodated easily by a model whose one-dimensional response is shown in Fig. 2, where at higher rate of strain the value of the yield stress jumps to a higher value along the vertical of the line  $AA_1A_2$ . Remarks concerning unloading from a point P at a higher rate of strain, say  $\dot{\epsilon}_2$ , discussed with reference to Fig. 1 are also applicable to unloading from points such as P in Fig. 2. Again, every point along PQ experiences the same amount of total strain and the same amount of plastic strain.

The response in one dimension of the "rate-dependent" material behavior of the type discussed in the preceding two paragraphs is once more depicted in Fig. 3, where again the curve AOB represents the "rate-independent" idealization. This model displays an increase in the linear range of the stress-strain response and, corresponding to different constant strain rates  $\dot{\epsilon}_1, \dot{\epsilon}_2$ , admits the so-called "dynamic yield stress" at points such as  $A_1, A_2$ . It is important to observe that in contrast to the model associated with Fig. 1, the model associated with Fig. 3 also implies accumulation of plastic strain beyond the "static yield" stress above point A. However, this does not appear to have been reported experimentally and may be even difficult to observe. An unloading line from a point P parallel to OA leads to a definition of plastic strain represented by the segment  $OR_2$ , which is unacceptable: Compare  $OR_2$  to the segment OR of Fig. 1, which for easy

comparison is also reproduced in Fig. 3. A model corresponding to the material response indicated in Fig. 3 has been used during loading to predict higher "dynamic yield" stresses at higher strain rates by different procedures. In particular, this feature of the "rate-dependent" behavior has been discussed by Malvern [18] and by Wilkins and Guinan [19] using a rate-independent yield function. More recently, Rubin [20] has proposed a different approach for predicting "dynamic yield" stress such as  $A_1, A_2$  in Fig. 3 with the use of a rate-dependent yield function along with a rate-independent constitutive equation for stress.

We now discuss constitutive equations for an elastic-viscoplastic material, which reflect properties for one-dimensional response discussed with reference to Fig. 1. Again, we adopt the strain space formulation of plasticity as primary, and admit the existence of a yield or loading function of the form (2.5). Keeping in mind the loading criteria of the strain space and the associated terminology in (2.6), we introduce constitutive equations for plastic strain and rate of work-hardening by the assumptions

$$\dot{\epsilon}_{KL}^p = \begin{cases} 0 & , \text{ during unloading or neutral loading } , & (a) \\ M_{KLMN} \dot{\epsilon}_{MN} & , \text{ during loading } , & (b) \end{cases} \quad (2.12)$$

and an expression of the form (2.7) which, in view of (2.12)<sub>2</sub>, may also be written as

$$\dot{\kappa} = \bar{C}_{KL} \dot{\epsilon}_{KL} . \quad (2.13)$$

In (2.12) and (2.13),  $M_{KLMN}$  and  $\bar{C}_{KL}$  are, respectively, the rectangular Cartesian components of a fourth order tensor and symmetric second order tensor functions  $\underline{M}$  and  $\bar{C}$  of the variables\* (2.3)<sub>2</sub>. As in the development

\*The arguments of the response functions  $\underline{M}$  and  $\bar{C}$  could include also the strain rate  $\dot{\epsilon}_{KL}$  but the forms of the response coefficient functions in (2.12)<sub>2</sub> and (2.13) suffice for our present discussion.



of the rate-independent theory (see Green and Naghdi [12]), it can be shown that the response coefficient in (2.13), can be expressed in the form

$$M_{KLMN} = \lambda \rho_{KL} \frac{\partial g}{\partial e_{MN}} , \quad (2.14)$$

or that equivalently  $\dot{e}_{KL}^p$  is again given by (2.6d) during loading. Also, the consistency condition is again of the same form as (2.9).

Next, it is convenient to introduce the constitutive equation for the stress response in the form

$$\begin{cases} s_{KL} = {}_1s_{KL} + {}_2s_{KLMN} \dot{e}_{MN}^p , & \text{during loading when } g=0, \hat{g} > 0 , & (a) \\ s_{KL} = {}_1s_{KL} , & \text{during unloading and neutral loading when } g=0, \hat{g} \leq 0 . & (b) \end{cases} \quad (2.15)$$

In (2.15), the second order tensor  ${}_1s_{KL}$  and the fourth order tensor  ${}_2s_{KLMN}$  satisfy obvious symmetries and each is regarded to depend on the variables<sup>\*</sup>  $u$ , i.e.,

$$s_1 = {}_1\hat{s}_{KL}(u) , \quad {}_2s_{KLMN} = {}_2\hat{s}_{KLMN}(u) . \quad (2.16)$$

The stress response characterized by (2.15) and (2.16) is linear in the strain rate through the constitutive equation for  $\dot{e}_{MN}^p$  during loading. During unloading and neutral loading, the stress response has the same form as that of the rate-independent theory; and hence, without loss in generality, we may identify  ${}_1\hat{s}_{KL}$  in (2.16)<sub>1</sub> with  $\hat{s}_{KL}$  of (2.3)<sub>1</sub>. Also the second part of the stress response during loading, namely  ${}_2\hat{s}_{KLMN} \dot{e}_{MN}^p$ , represents the jump in  $s_{KL}$  from a point such as Q (see Fig. 1) on the loading function and will assume different values depending on the rate of strain  $\dot{e}_{MN}$ .

<sup>\*</sup>Again the strain rate  $\dot{e}_{KL}$  could be included in the argument of  ${}_2s_{KLMN}$  but this is unnecessary in the present discussion.

In the next section we restrict attention to a special form of the stress response (2.15) given by

$$\begin{cases} s_{KL} = {}_1L_{KLMN}(e_{MN} - e_{MN}^p) + {}_2L_{KLMN}\dot{e}_{MN}^p & , \text{ when } g = 0, \hat{g} > 0 & , \quad (a) \\ s_{KL} = {}_1L_{KLMN}(e_{MN} - e_{MN}^p) & , \text{ when } g = 0, \hat{g} \leq 0 & , \quad (b) \end{cases} \quad (2.17)$$

where  ${}_1L_{KLMN}$  and  ${}_2L_{KLMN}$  are the rectangular Cartesian components of constant fourth order tensors  ${}_1\bar{L}$  and  ${}_2\bar{L}$ , respectively.

### 3. Restrictions on constitutive equations for plastic strain rate and the stress response (2.17)

We first recall in this section the primitive work assumption introduced by Naghdi and Trapp [9] leading to the work inequality (3.2) and then discuss the nature of the restrictions which can be placed on the constitutive equations for the plastic strain rate and the special response (2.17).

Consider a closed cycle of a spatially homogeneous motion in the closed time interval<sup>\*</sup>  $[t_1, t_2]$ ,  $(t_1 < t_2)$ . The cycle is said to be smooth if the time derivatives of displacement, strain and associated kinematical quantities are continuous in  $[t_1, t_2]$  and assume the same values for each material point at times  $t_1$  and  $t_2$ . We designate such a smooth spatially homogeneous closed cycle of deformation by  $C(t_1, t_2)$  and recall from [9] the following work assumption: The external work done on the body by surface tractions and by body forces in any smooth spatially homogeneous closed cycle is nonnegative, i.e.,

$$\int_{t_1}^{t_2} \left[ \int_{\partial R_0} p_{iK} N_K v_i dA + \int_{R_0} \rho_0 b_i v_i dV \right] dt \geq 0, \quad (3.1)$$

for all cycles  $C(t_1, t_2)$ . In (3.1),  $R_0$  is the region of space occupied by the body in its reference configuration,  $\partial R_0$  is the closed boundary surface of  $R_0$ ,  $\rho_0$  is the mass density in the reference configuration,  $v_i$  are the components of the velocity,  $b_i$  are the components of the body force per unit mass,  $N_K$  are the components of the outward unit normal to  $\partial R_0$ ,  $p_{iK} N_K$  represent the components of the stress vector measured per unit area in the reference configuration, and  $dA$  and  $dV$  refer to elements of area and volume in the reference configuration.

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\* Recall that a homogeneous motion is one whose deformation gradient is independent of the material coordinates so that, in a spatially homogeneous motion, the strain tensor  $\underline{E}$  is a function of time only. For a closed spatially homogeneous cycle in the closed time interval  $[t_1, t_2]$ , the displacement  $\underline{x}$  and the strain  $\underline{E}$  assume the same values at times  $t_1$  and  $t_2$ .

It is shown in [9] that for any cycle  $C(t_1, t_2)$  the assumption (3.1) leads to\*\*

$$\int_{t_1}^{t_2} s_{KL} \dot{e}_{KL} dt \geq 0 \quad (3.2)$$

As already noted in section 1, the above work inequality is valid for rate-dependent materials even though originally it was used in [9] for rate-independent elastic-plastic solids.

Let  $\underline{E}^0$  with rectangular Cartesian components  $e_{KL}^0$  refer to an existing state of strain inside a loading surface in the six-dimensional strain space such that  $g(e_{KL}^0, e_{KL}^p, \kappa) < 0$  at time  $t_1$ . Let  $\bar{t}$  designate the first occurrence of plastic strain at which time  $g(e_{KL}^Y, e_{KL}^p, \kappa) = 0$ , where  $e_{KL}^Y$  denotes the value of the strain at a point on the loading surface. We note that it is always possible to find a path inside  $g(\underline{E}^Y, \underline{E}^p, \kappa) = 0$  from  $\underline{E}^0$  to  $\underline{E}^Y$  such that the strain rate is any desired value at  $\underline{E}^Y$ . Consider now a closed spatially homogeneous strain cycle starting and ending at  $\underline{E}^0$ . From  $\underline{E}^0$  to  $\underline{E}^Y$  the loading takes place elastically during the interval  $t_1 \leq t < \bar{t}$  and continues in the viscoplastic range in the interval  $\bar{t} \leq t < \bar{t} + \frac{1}{n}$  at a constant strain rate  $\underline{M} (= \dot{\underline{E}})$ . At the end of this interval (corresponding to a point such as P in Fig. 1) the strain has a value  $\underline{E}^Y + \frac{1}{n} \underline{M}$ . This is followed by a constant deceleration process to a state of zero strain rate (corresponding to points such as  $Q_1, Q_2$  on the rate-independent response curve in Fig. 1). Such deceleration processes are assumed to occur at a decreasing rate of strain specified by the constant rate  $-\underline{\ell} \underline{M} (= \ddot{\underline{E}})$  during the time interval  $\bar{t} + \frac{1}{n} \leq t < \bar{t} + \frac{1}{n} + \frac{1}{\underline{\ell}}$  while

\*\* An inequality similar in form to (3.2), but with limitation to infinitesimal deformation, is the starting point of Il'iushin's discussion in [5] and is referred to by him as the "postulate of plasticity."

§ Recall this if the path in strain space is parametrized with respect to time  $t$ , then the strain rate  $\dot{\underline{E}}$  is directed along the tangent to the path. If  $\dot{\underline{E}} = \text{const.}$ , then the path is a straight line.

loading continues. At the end of this time interval, the strain rate vanishes and the strain has the value  $\bar{\epsilon}^Y + (\frac{1}{n} + \frac{1}{2\ell})\bar{M}$ . A sequence of such decelerated processes (corresponding to paths  $PQ_1, PQ_2, \dots$ , in Fig. 1), in the limit as  $\ell \rightarrow \infty$ , results in a rapid path (corresponding to  $PQ$  in Fig. 1) to a state of zero strain rate with the value of strain being  $\bar{\epsilon}^Y + \frac{1}{n}\bar{M}$  (corresponding to the value of strain at  $Q$  in Fig. 1). Thereafter, unloading takes place during the time interval  $\bar{t} + \frac{1}{n} + \frac{1}{\ell} \leq t < t_2$  with the strain returning to  $\bar{\epsilon}(t) = \bar{\epsilon}^0$ . The strain cycle just described may be summarized as follows:

$$\bar{\epsilon}(t) = \begin{cases} \bar{\epsilon}^0, & t < t_1, & (a) \\ \bar{\epsilon}^0 \rightarrow \bar{\epsilon}^Y, \quad \dot{\bar{\epsilon}} \rightarrow \bar{M} \text{ at } \bar{\epsilon}^Y, & t_1 \leq t < \bar{t}, & (b) \\ \bar{\epsilon}^Y + \bar{M}(t - \bar{t}), & \bar{t} \leq t < \bar{t} + \frac{1}{n}, & (c) \\ \bar{\epsilon}^Y + [\frac{1}{n} + (t - \bar{t} - \frac{1}{n}) - \frac{\ell}{2}(t - \bar{t} - \frac{1}{n})^2]\bar{M}, & \bar{t} + \frac{1}{n} \leq t < \bar{t} + \frac{1}{n} + \frac{1}{\ell}, & (d) \\ \bar{\epsilon}^Y \rightarrow \bar{\epsilon}^0, \quad \dot{\bar{\epsilon}} = \bar{M} \text{ at } \bar{\epsilon}^Y, & \bar{t} + \frac{1}{n} + \frac{1}{\ell} \leq t < t_2, & (e) \\ \bar{\epsilon}^0, & t_2 \leq t. & (f) \end{cases} \quad (3.3)$$

The one-dimensional version of the above strain cycle is indicated in Fig. 4 by the cycle  $C_1 Q P P' Q' C_6$ : The point  $C_1$  at the strain  $\bar{\epsilon}^0$  in Fig. 4 corresponds to (3.3a); the elastic loading path  $C_1 Q$  corresponds to (3.3b); after the jump in stress at  $Q$ , the path  $PP'$  at a constant strain rate  $\bar{M}$  corresponds to (3.3c); the path  $P' \bar{Q}$ , which in the limit of rapid deceleration becomes  $P' Q'$ , corresponds to (3.3d); the elastic unloading path  $Q' C_6$  corresponds to (3.3e); and finally the point  $C_6$  at the strain  $\bar{\epsilon}^0$  corresponds to (3.3f).

With the use of the special stress response (2.17), application of the work inequality (3.2) to the strain cycle (3.3) leads to

$$\int_{t_1}^{t_2} 1^{L_{KLMN}} (\bar{e}_{MN} - e_{MN}^P) \dot{\bar{e}}_{KL} dt + \int_{\bar{t}}^{\bar{t} + (\frac{1}{n} + \frac{1}{\ell})} 2^{L_{KLMN}} \dot{\bar{e}}_{MN}^P \dot{\bar{e}}_{KL} dt \geq 0. \quad (3.4)$$

\* Note that if  $\bar{\epsilon}^Y + (\frac{1}{n} + \frac{1}{2\ell})\bar{M}$  is close to  $\bar{\epsilon}^Y$ , then the strain  $\bar{\epsilon}^0$  will remain inside the yield surface since the loading function is continuous in  $\bar{\epsilon}$ .

where use has been made of the fact that  $\dot{\underline{E}}^P$  is nonzero only during the time interval  $[\bar{t}, \bar{t} + (\frac{1}{n} + \frac{1}{\ell})]$ . The strain cycle (3.3) begins and ends at the constant strain  $\underline{E}^0$ . By writing  $\dot{\underline{E}}$  as  $(\underline{E} - \underline{E}^0)$  and the use of the fact that  $\underline{E}(t) = \underline{E}^0$  at  $t_1$  and  $t_2$ , the first integral on the left-hand side of the above inequality can be integrated by parts and (3.4) reduces to

$$\int_{\bar{t}}^{\bar{t} + (\frac{1}{n} + \frac{1}{\ell})} {}_1L_{KLMN}(\underline{e}_{KL} - \underline{e}_{KL}^0)\dot{\underline{e}}_{MN}^P dt + \int_{\bar{t}}^{\bar{t} + (\frac{1}{n} + \frac{1}{\ell})} {}_2L_{KLMN}\dot{\underline{e}}_{KL}\dot{\underline{e}}_{MN}^P dt \geq 0 \quad (3.5)$$

After estimating the various integrals on the left-hand side of (3.5) by using the Taylor series expansion of the integrals about  $t = \bar{t} + \frac{1}{n}$  and  $t = \bar{t}$  and allowing in the limit  $\ell \rightarrow \infty$ , i.e., the "slowing down period," to take place very fast and approach a rapid path (corresponding to  $P'Q'$  in Fig. 4), the work inequality (3.5) yields (for details see Appendix A)

$$(\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{M}^Y \underline{M} + \underline{M} \cdot {}_2\underline{L} \underline{M}^Y \underline{M} \geq 0 \quad (3.6)$$

which in component form reads

$$(\underline{e}_{KL}^Y - \underline{e}_{KL}^0) {}_1L_{KLMN} \underline{M}_{MNRS}^Y \underline{M}_{RS} + \underline{M}_{KL} {}_2L_{KLMN} \underline{M}_{MNRS}^Y \underline{M}_{RS} \geq 0 \quad (3.6a)$$

where for the inner product of any two tensors  $\underline{A}, \underline{B}$  we have used the notation  $\underline{A} \cdot \underline{B} = \text{trace } \underline{A}^T \underline{B}$ ,  $\underline{M}^Y$  is the value at  $\underline{E}^Y$  of the fourth order tensor  $\underline{M}$  whose components  $\underline{M}_{MNRS}$  are given by (2.14) and  $\underline{M}_{RS}$  are the components of the strain rate  $\underline{M}$  which occurs in the strain cycle (3.3). The inequality (3.6) or (3.6a) is a necessary condition for the validity of (3.5) and hence also (3.2).

Now let  $\underline{E}^0$  approach  $\underline{E}^Y$  while maintaining the strain rate fixed at  $\bar{t}$ . It then follows from (3.6a) that

$${}_2L_{KLMN}\dot{\underline{e}}_{KL}\dot{\underline{e}}_{MN}^P > 0 \quad (3.7)$$

Since (3.6) must hold for all  $\underline{M}$  and the coefficients of  $\underline{M}$  in (3.6) are

independent of rates, we may also deduce that

$${}_1L_{KLMN}(e_{KL}^Y - e_{KL}^O)\dot{e}_{MN}^P \geq 0 \quad . \quad (3.8)$$

The two inequalities (3.7) and (3.8) are both necessary and sufficient conditions for the validity of the coupled inequality (3.6). Further, by a special but arbitrary choice of  $\underline{E}^O$  such that the path  $\underline{E}^O$  to  $\underline{E}^Y$  is traversed at a constant strain rate, (3.8) can be reduced to

$${}_1L_{KLMN}\dot{e}_{KL}\dot{e}_{MN}^P \geq 0 \quad . \quad (3.9)$$

The inequalities (3.7)-(3.9) are restrictions on the constitutive equations of a finitely deformed elastic-viscoplastic material with the special stress response (2.17). These restrictions hold in all motions, even though they have been deduced from consideration of homogeneous motions alone. We note that the results (3.7)-(3.9) include as a special case those relevant to the rate-independent theory of plasticity. In fact, (3.8) and (3.9) are identical to those derived previously by Naghdi and Trapp [10, Eqs. (30) and (31)].

#### 4. Further implications of the inequalities obtained in Section 3

We discuss now certain features of the restrictions (3.7)-(3.9). As shown in [11], it follows from (3.9) that

$$\lambda \, {}_1 L_{KLMN}^{\rho MN} = \gamma_1 \frac{\partial g}{\partial e_{KL}} , \quad (\gamma_1 > 0) , \quad (4.1)$$

where  $\gamma_1 = \gamma_1(u)$  is a positive scalar independent of rates. Similarly, from (3.7), it can be deduced that

$$\lambda \, {}_2 L_{KLMN}^{\rho MN} = \gamma_2 \frac{\partial g}{\partial e_{KL}} , \quad (\gamma_2 > 0) , \quad (4.2)$$

where  $\gamma_2 = \gamma_2(u)$  is a positive scalar independent of rates. The left-hand sides of both (4.1) and (4.2) involve rate of plastic strain (see Eq. (2.6a)). Moreover, these quantities (on the left-hand sides of (4.1) and (4.2)) are directed along the normal to the yield (or loading) surface in strain space. Next, by (3.8) and the fact that  $\gamma_1 > 0$ , we have

$$(e_{KL}^y - e_{KL}^o) \frac{\partial g}{\partial e_{KL}} > 0 \quad (4.3)$$

and this implies convexity of the yield (or loading) surface in strain space.

We now examine the implication of the restrictions (4.1) and (4.2), which are obtained from (3.7) and (3.9), on the constitutive equation (2.17a). Since  $\gamma_1$  and  $\gamma_2$  are nonzero, by (4.1) and (4.2) we have

$$\frac{\lambda}{\gamma_1} {}_1 L_{KLMN}^{\rho MN} = \frac{\lambda}{\gamma_2} {}_2 L_{KLMN}^{\rho MN} . \quad (4.4)$$

Now a part of the stress response in (2.17a) which involves  ${}_2 \dot{L}$ , with the help of (2.12b), (2.14) and (4.4), can be written as

$$\begin{aligned} {}_2 L_{KLMN}^{\rho MN} \dot{e}_{MN}^p &= {}_2 L_{KLMN}^{\lambda \rho MN} \hat{g} \\ &= \frac{\gamma_2}{\gamma_1} {}_1 L_{KLMN}^{\lambda \rho MN} \hat{g} \\ &= \gamma_1 L_{KLMN}^{\rho MN} \dot{e}_{MN}^p , \end{aligned} \quad (4.5)$$



where we have set

$$\gamma = \frac{\gamma_2}{\gamma_1} . \quad (4.6)$$

Since  $\gamma_1$  and  $\gamma_2$  may be determined from (4.1) and (4.2), the coefficient  $\gamma$  defined by (4.6) can be regarded as known and of course may be taken to be a constant. The conclusion (4.5)<sub>3</sub> enables us to rewrite the stress response (2.17a) in the form

$$s_{KL} = {}_1L_{KLMN}[(e_{MN} - e_{MN}^p) + \gamma {}_2L_{KL} g] \quad (4.7)$$

and this has the advantage of containing only the constitutive coefficient  ${}_1L$ .

The inequality (3.5) when appropriately specialized to the strain cycle corresponding to  $C_1 QPP' Q' C_6$  in Fig. 4 can be expressed as

$$\int_{t_1}^{t_2} (e_{KL}^y - e_{KL}^o) {}_1L_{KLMN} \dot{e}_{MN}^p dt + \int_{t_1}^{t_2} (e_{KL} - e_{KL}^y) {}_1L_{KLMN} \dot{e}_{MN}^p dt + \int_{t_1}^{t_2} ({}_2L_{KLMN} \dot{e}_{MN}^p) \dot{e}_{KL} dt \geq 0 , \quad (4.8)$$

and in one-dimension (using the notations of sections 1 and 2) has the form

$$\int_{C(e)} (e^y - e^o) {}_1L de_p + \int_{C(e)} (e - e^y) {}_1L de_p + \int_{C(e)} ({}_2L \dot{e}^p) de \geq 0 . \quad (4.9)$$

In (4.9), the relevant material coefficients for convenience have been replaced by the constants  ${}_1L$  and  ${}_2L$ , the quantity  $({}_2L \dot{e}^p)$  in the integrand of the third integral represents the jump in stress from  $Q$  to  $P$  and  $C(e)$  refers to a closed strain cycle such as  $C_1 QPP' Q' C_6$  in Fig. 4. The shaded areas enclosed by  $C_1 QPP' Q' C_6$  consist of (i) the area of the parallelogram  $C_6 C_1 QD$ , (ii) the area of the triangle  $QQ'D$  and (iii) the area enclosed by  $PP' QQ'$ . It is easily seen that these areas represent the three integrals (from left to right) on the left-hand side of (4.9).

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### Appendix A

The purpose of this appendix is to provide some of the mathematical details used in obtaining the estimates for the integrals in the work inequality (3.5). Most of the calculations are similar to corresponding developments carried out in [11] with the use of more general stress constitutive equations, but there are also some differences in the results obtained here. Previously in [11], the closed strain cycle in  $E$  was so constructed that the path from  $\underline{E}^0$  to  $\underline{E}^Y$ , as well as the reverse path from  $\underline{E}^Y$  to  $\underline{E}^0$ , are both traversed at an arbitrary but the same constant strain rate. This stipulation was necessary in [11] in order to effect explicit estimates for certain integrals. In the present paper, the path from  $\underline{E}^0$  to  $\underline{E}^Y$  and the reverse path are not necessarily traversed at constant strain rates. Nevertheless, explicit estimates for the integrals over these paths are possible because the response coefficient  $\gamma_L$  in the special constitutive equation (2.17) is a constant tensor.

Let  $G(s)$  defined by

$$G(s) = \int_{s_0}^s h(t) dt \quad (A1)$$

be continuous and at least twice differentiable in the interval  $s_0 \leq t \leq s$  and, for later convenience, put

$$h(t) = f(t)g(t) \quad (A2)$$

Then, the Taylor series expansion of (A1) about  $t = s_0$  is

$$\begin{aligned}
G(s) = & (s-s_0)f(s_0)g(s_0) + \frac{1}{2!} (s-s_0)^2 [f'(s_0)g(s_0) + f(s_0)g'(s_0)] \\
& + \frac{1}{3!} (s-s_0)^3 [f''(s_0)g(s_0) + 2f'(s_0)g'(s_0) + f(s_0)g''(s_0)] \\
& + O((s-s_0)^4) ,
\end{aligned} \tag{A3}$$

where  $O$  is the usual order symbol and prime denotes derivative with respect to  $t$ .

In order to indicate the manner in which the integrals in (3.5) can be estimated about  $\bar{t}$ , consider first the integral

$$I_1 = \int_{\bar{t}}^{\bar{t} + \frac{1}{n}} (\underline{E} - \underline{E}^0) \cdot {}_1\underline{L} \dot{\underline{E}}^P dt , \tag{A4}$$

where the integrand of (A4) also occurs in the first integral in (3.5) and the notation for the inner product of any two tensors is defined following (3.6a). After substituting the relevant value of  $\underline{E}$  from (3.3) and applying the Taylor expansion (A3) of the integral (A1) to (A4) with

$f(t) = [\underline{E}^Y - \underline{E}^0 + \underline{M}(t-\bar{t})]$  and  $g(t) = {}_1\underline{L} \dot{\underline{E}}^P$ , (A4) becomes

$$\begin{aligned}
I_1 &= \int_{\bar{t}}^{\bar{t} + \frac{1}{n}} [\underline{E}^Y - \underline{E}^0 + \underline{M}(t-\bar{t})] \cdot {}_1\underline{L} \dot{\underline{E}}^P dt \\
&= \left[ \frac{1}{n} (\underline{E}^Y - \underline{E}^0) + \frac{1}{2n^2} \underline{M} \right] \cdot {}_1\underline{L} \dot{\underline{E}}^{PY} + \left[ \frac{1}{2n^2} (\underline{E}^Y - \underline{E}^0) + \frac{1}{3n^3} \underline{M} \right] \cdot {}_2\underline{L} \ddot{\underline{E}}^P \\
&= \frac{1}{n} (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{M}^Y \underline{M} + \frac{1}{2n^2} [\underline{M} \cdot {}_1\underline{L} \underline{M}^Y \underline{M} + (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{N}^Y \underline{M}] + O\left(\frac{1}{n^3}\right) , \tag{A5}
\end{aligned}$$

where in writing (A5)<sub>3</sub> use is made of the fourth order tensor  $\underline{M}$  defined following (2.12)<sub>2</sub>, the superscript  $Y$  denotes the value of a function on the yield surface at  $t = \bar{t}$  and where the fourth order tensor  $\underline{N}$  stands for the abbreviation

$$\underline{N} = N(u, \underline{M}) = \frac{\partial \underline{M}}{\partial \underline{E}} \cdot \underline{M} + \frac{\partial \underline{M}}{\partial \underline{E}^P} \cdot \underline{M} \underline{M} + \frac{\partial \underline{M}}{\partial \underline{\kappa}} \underline{C} \cdot \underline{M} \underline{M} . \tag{A6}$$

Similarly, consider the integral

$$I_2 = \int_{\bar{t}}^{\bar{t} + \frac{1}{n}} \dot{\underline{E}} \cdot {}_2\underline{L} \dot{\underline{E}}^p dt \quad (A7)$$

whose integrand also occurs in the second integral in (3.5). Substituting for the strain rate from (3.3c) and applying the Taylor expansion (A3) of the integral (A1) to (A7), we obtain

$$I_2 = \frac{1}{n} \underline{M} \cdot [{}_2\underline{L} \underline{M}^Y \underline{M} + \frac{1}{2n} {}_2\underline{L} \underline{N}^Y \underline{M}] + O(\frac{1}{n^3}) \quad (A8)$$

Now by a procedure similar to that discussed above, we may obtain an estimate for the two integrals in (3.5). Thus, the Taylor expansion of the first integral in (3.5) about  $\bar{t} + \frac{1}{n}$  yields

$$\begin{aligned} I_3 &= \int_{\bar{t} + \frac{1}{n}}^{\bar{t} + \frac{1}{n} + \frac{1}{\ell}} (\underline{E} - \underline{E}^0) \cdot {}_1\underline{L} \dot{\underline{E}}^p dt \\ &= \frac{1}{2\ell} (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L}^{(M)} \bar{t} + \frac{1}{n} \underline{M} \\ &\quad + \left( \frac{1}{2n\ell} + \frac{1}{8\ell^2} \right) \underline{M} \cdot {}_1\underline{L}^{(M)} \bar{t} + \frac{1}{n} \underline{M} \\ &\quad - \frac{3}{8\ell^2} (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L}^{(N)} \bar{t} + \frac{1}{n} \underline{M} \\ &\quad + O\left(\frac{1}{n\ell^2}, \frac{1}{\ell^3}\right) \quad (A9) \end{aligned}$$

When the right-hand side of (A9)<sub>2</sub> is further expanded about  $\bar{t}$ , we obtain

$$\begin{aligned} I_3 &= \frac{1}{2\ell} (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{M}^Y \underline{M} + \frac{1}{2n\ell} \underline{M} \cdot {}_1\underline{L} \underline{M}^Y \underline{M} \\ &\quad + \frac{1}{2n\ell} (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{N}^Y \underline{M} \\ &\quad - \frac{3}{8\ell^2} (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{N}^Y \underline{M} \\ &\quad + O\left(\frac{1}{n^2\ell}, \frac{1}{n\ell^2}, \frac{1}{\ell^3}\right) \quad (A10) \end{aligned}$$

The estimate for the second integral on the left-hand side of (3.5) can be effected similarly and leads to

$$\begin{aligned}
 I_4 &= \int_{\frac{1}{t} + \frac{1}{n}}^{\frac{1}{t} + \frac{1}{n} + \frac{1}{\ell}} \dot{\underline{E}} \cdot {}_2\underline{L} \dot{\underline{E}}^p dt \\
 &= \frac{1}{3\ell} \underline{M} \cdot {}_2\underline{L} \underline{M}^Y \underline{M} \\
 &\quad + \left( \frac{1}{3n\ell} + \frac{1}{15\ell^2} \right) \underline{M} \cdot {}_2\underline{L} \underline{N}^Y \underline{M} \\
 &\quad + O\left( \frac{1}{n^2\ell}, \frac{1}{n\ell^2}, \frac{1}{\ell^3} \right). \quad (A11)
 \end{aligned}$$

Substituting the estimates (A5), (A8), (A10) and (A11) into (3.5) we finally obtain

$$\begin{aligned}
 &\left( \frac{1}{n} + \frac{1}{2\ell} \right) (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{M}^Y \underline{M} \\
 &\quad + \left( \frac{1}{2n^2} - \frac{3}{8\ell^2} \right) (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{N}^Y \underline{M} \\
 &\quad + \left( \frac{1}{2n^2} + \frac{1}{2n\ell} - \frac{3}{8\ell^2} \right) (\underline{E}^Y - \underline{E}^0) \cdot {}_1\underline{L} \underline{N}^Y \underline{M} \\
 &\quad + \left( \frac{1}{n} + \frac{1}{3\ell} \right) \underline{M} \cdot {}_2\underline{L} \underline{M}^Y \underline{M} + \left( \frac{1}{2n^2} + \frac{1}{3n\ell} + \frac{1}{15\ell^2} \right) \underline{M} \cdot {}_2\underline{L} \underline{N}^Y \underline{M} \\
 &\quad + O\left( \frac{1}{n^2\ell}, \frac{1}{n\ell^2}, \frac{1}{n^3}, \frac{1}{\ell^3} \right) \geq 0. \quad (A12)
 \end{aligned}$$

The inequality (A12) must hold for arbitrary  $\underline{M}$ , for all values of the parameters  $n, \ell$  and for all  $\underline{E}^0$ . As in [11], we consider now a sequence of strain cycles such that  $n, \underline{M}, \underline{E}^0$  are fixed while the "slowing down" period takes place faster and faster as  $\ell$  becomes larger and larger. In the limit, the sequence approaches the cycle involving the rapid path during deceleration and, as  $\ell \rightarrow \infty$ , (A12) reduces to

$$\frac{1}{n} (\underline{E}^Y - \underline{E}^0) \cdot \underline{1} \underline{L} \underline{M}^Y \underline{M} + \frac{1}{n} \underline{M} \cdot \underline{2} \underline{L} \underline{M}^Y \underline{M} + O\left(\frac{1}{n^2}\right) > 0 \quad . \quad (A13)$$

Next, consider a second sequence of strain cycles each member of which has a similar rapid path of deceleration. For fixed values of  $\underline{M}$  and  $\underline{E}^0$  but with progressively larger values of  $n$ , we may apply (A13) to this second sequence of cycles. Then, after multiplying (A13) by  $n$  and taking the limit as  $n \rightarrow \infty$ , we deduce the inequality (3.6).

### Captions for figures

- Fig. 1      Idealized mechanical response of an elastic-viscoplastic material at various strain rates. The curves  $OAB_1$  and  $OAB_2$  represent the material response at higher rates of strain associated with "rate-dependent" model, while the curve  $OAB$  represents the "rate-independent" response and may be viewed as representing the material response when the strain rate approaches zero. The dashed line curves  $PQ_1, PQ_2, \dots, PQ$  represent unloading with various degrees of acceleration (the slowest being  $PQ_1$ ) from a point  $P$  on curve  $OAB_2$ . The instantaneous unloading from  $P$  takes place along the rapid path  $PQ$ . Also shown are elastic unloading from a point  $Q$  and the identification of plastic strain by  $OR$  along the  $e$ -axis. The values of the total strain and the plastic strain are the same for every point along the dashed curve  $PQ$ .
- Fig. 2      An idealized one-dimensional response of a "rate-dependent" model with most of its features being similar to that depicted in Fig. 1, but with a different characteristic for the initial response at various rates of strain.
- Fig. 3      An idealized one-dimensional response of a "rate-dependent" model displaying an increase in the linear range of the stress-strain response. Corresponding to different constant strain rates, the model exhibits "dynamic yield stress" at points such as  $A_1$  and  $A_2$ . Also shown is the curve  $OAQB$  representing the "rate-independent" response.



Fig. 4 One-dimensional version of a strain cycle  $C_1 Q P P' Q' C_6$  corresponding to the cycle of homogeneous strain defined by (3.3) and used in the calculation of the work inequality (3.5). Also, the three integrals in (4.8) represent the (cross-hatched) areas enclosed by  $C_1 Q P P' Q' C_6$ .

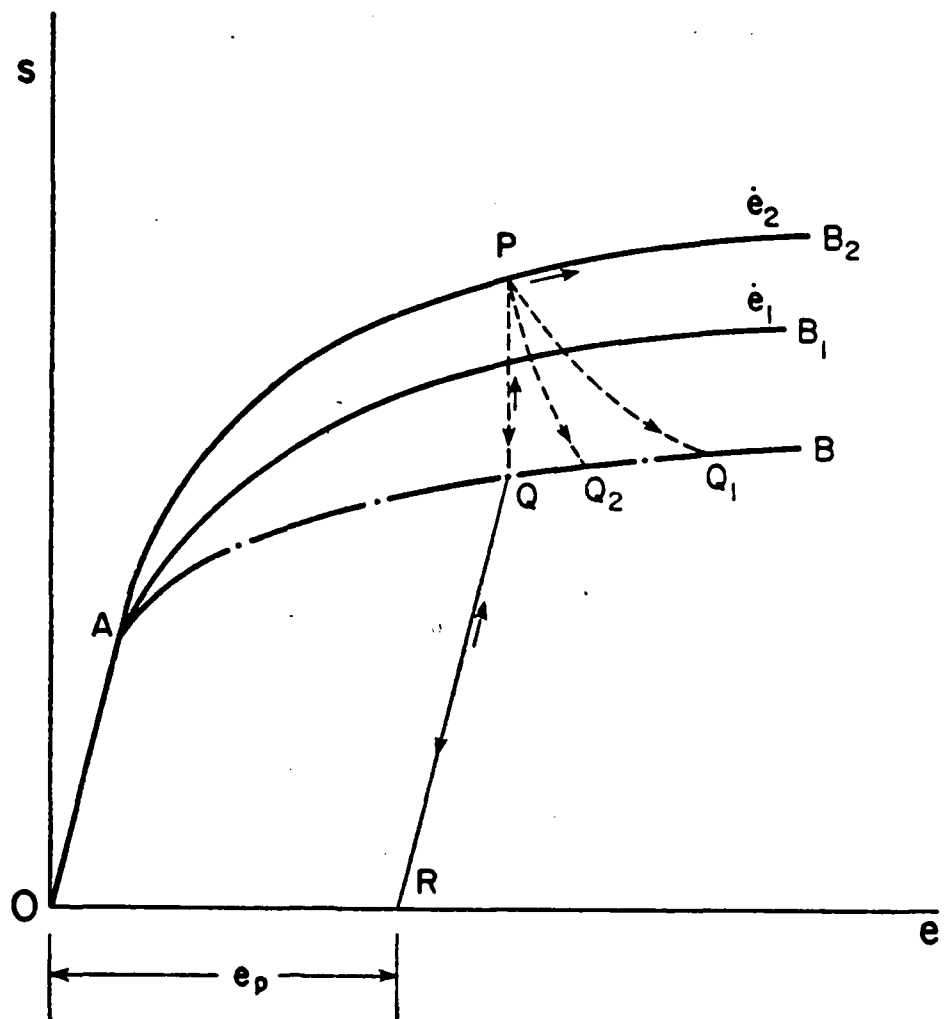


Fig. 1

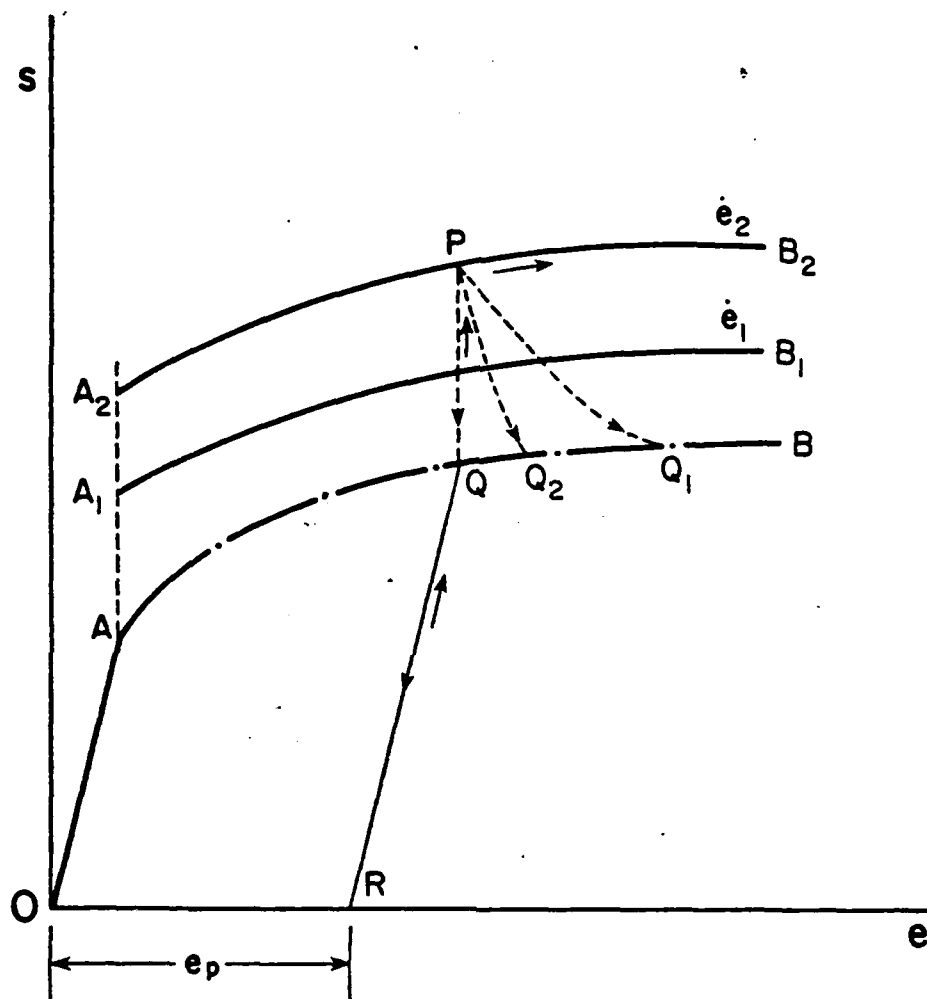


Fig. 2

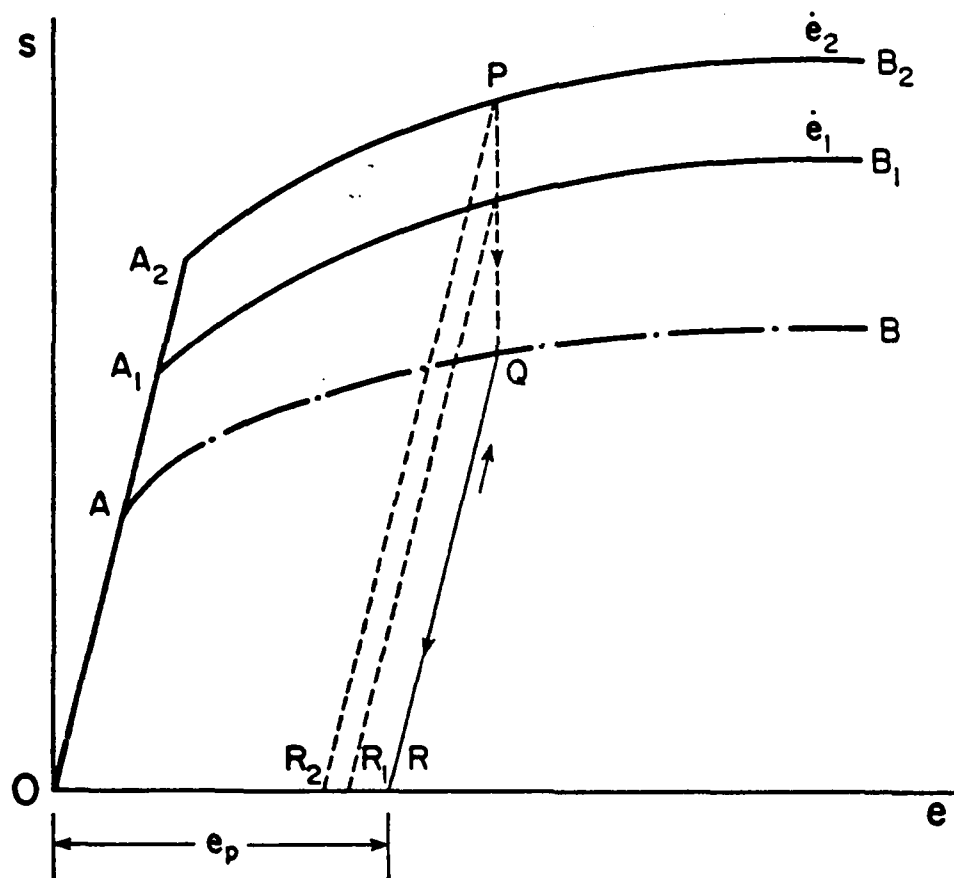


Fig. 3

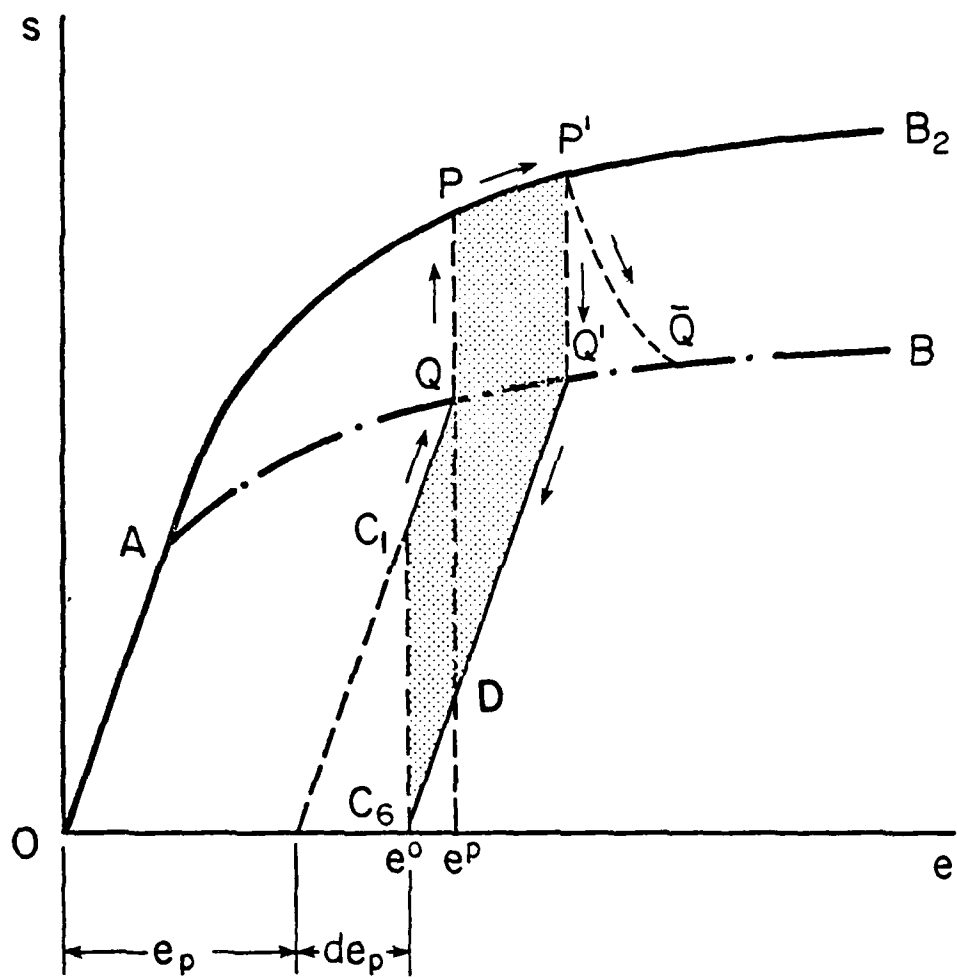


Fig. 4

